Fix a lattice $M \cong \mathbb{Z}^n$ $\quad M_{\mathbb{R}} = M \otimes \mathbb{R}$

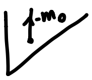
Fix a strictly polyhedral cone $\sigma \subseteq M_{\mathbb{R}}$ $\quad N = \mathrm{Hom}(M, \mathbb{Z})$

$P = \sigma \cap M$ monoid $\quad R = \widehat{k[P]}$ completion w.r.t. $\mathfrak{m}$

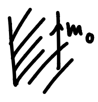
$\mathfrak{m} \subseteq k[P]$ max. monomial ideal

<u>Def</u>: A <u>wall</u> in $M_{\mathbb{R}}$ is a pair (d, f_d) where

① $d \subseteq M_{\mathbb{R}}$ is a convex rat^l polyhed^l codim 1 cone with an element $m_0 \in P \setminus \{0\}$ s.t. $d = d - \mathbb{R}_{\geq 0} m_0$



The wall is <u>incoming</u> if $d = d + \mathbb{R}_{\geq 0} m_0$, otherwise <u>outgoing</u>



② $f_d = 1 + \sum_{k \geq 0} c_k z^{k m_0} \in R$

A scattering diagram $\mathfrak{D}$ is a collection of walls s.t.

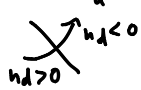
$$\#\{(d, f_d) \in \mathfrak{D} \mid f_d \not\equiv 1 \bmod \mathfrak{m}^k\} < \infty$$

is finite for any $k > 0$

$\mathrm{Supp}\,\mathfrak{D} = \bigcup_{(d, f_d) \in \mathfrak{D}} d$ $\quad \mathrm{Sing}\,\mathfrak{D}$ = locus where $\mathrm{Supp}\,\mathfrak{D}$ is <u>not</u> a mfd.

For a path $\gamma : [0,1] \to M_{\mathbb{R}} \setminus \mathrm{Sing}\,\mathfrak{D}$ with endpoints not in $\mathrm{Supp}\,\mathfrak{D}$ define the path-ordered product $\theta_{\gamma, \mathfrak{D}} \in \mathrm{Aut}(R)$ as follows

A single crossing of the wall (d, f_d) gives the automorphism $z^m \mapsto z^m f_d^{\langle n_d, m \rangle}$ where $n_d \in N$ annihilates d, is primitive & decreasing on γ



All these automorphisms are elements of the pro-nilpotent Lie grp w. Lie algebra

$$\mathfrak{g} = \bigoplus_{m \in P \setminus \{0\}} z^m (m^\perp \otimes k) \qquad m^\perp \subseteq N$$

$$z^m n \leftrightarrow z^m \partial_n$$

$z^m \partial_n z^{m'} = \langle n, m' \rangle z^{m+m'}$

This vector field preserves $\bigwedge_i d\log z^{e_i}$ $\quad e_1 \ldots e_n$ basis of M

$G = \exp \mathfrak{g}$

Fix a codim 2 sublattice $\Lambda \subseteq M$

Have $\mathfrak{h}_\Lambda, \mathfrak{h}_\Lambda^\perp\ \mathfrak{h}_\Lambda^{\parallel} \subseteq \mathfrak{g}$

$$\mathfrak{h}_\Lambda = \bigoplus_{m \in P \setminus \{0\}} z^m (m^\perp \cap \Lambda^\perp) \otimes k$$

$$= \mathfrak{h}_\Lambda^\perp \oplus \mathfrak{h}_\Lambda^{\parallel} \quad \text{where}$$

$$\mathfrak{h}_\Lambda^\perp = \bigoplus_{m \in P \setminus \Lambda} z^m (m^\perp \cap \Lambda^\perp) \otimes k$$

$$\mathfrak{h}_\Lambda^{\parallel} = \bigoplus_{m \in P \cap \Lambda} z^m \Lambda^\perp \otimes k$$

$$[z^m \partial_n, z^{m'} \partial_{n'}] = \langle n, m' \rangle z^{m+m'} \partial_{n'} - \langle n', m \rangle z^{m+m'} \partial_n$$

This implies $[\,,]$ is zero on $\mathfrak{h}_\Lambda^{\parallel}$ and $[\mathfrak{h}_\Lambda^{\parallel}, \mathfrak{h}_\Lambda^\perp] \subseteq \mathfrak{h}_\Lambda^\perp$

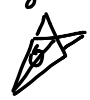
$H_\Lambda = \exp(\mathfrak{h}_\Lambda)$ $\quad H_\Lambda^{\parallel} = H_\Lambda / H_\Lambda^\perp$

<u>Theorem</u> (KS dim 2 04, GS dim 2 07)

Let $\mathfrak{D}$ be a scattering diagram such that for any dim $n-2$ cell (joint) j of $\mathrm{Sing}\,\mathfrak{D}$ and for any suff. small loop γ around j

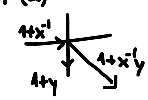
$\theta_{\gamma, \mathfrak{D}}$ has trivial image in $H_{\Lambda_j}^{\parallel}$ where

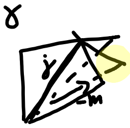
Λ_j = integral tangent vectors parallel to j



Then there exists a scattering diagram $S(\mathfrak{D}) \supset \mathfrak{D}$ such that $S(\mathfrak{D}) \setminus \mathfrak{D}$ consists only of outgoing walls and

$\theta_{\gamma, S(\mathfrak{D})} = \mathrm{id}$ $\quad \forall$ loops γ





$\supset j - \mathbb{R}_{\geq 0} m$

$\exp(z^m \partial_n)$

<u>Special case</u>

$\{\,,\} : N \times N \to \mathbb{Z}$ skew symmetric form $e_1 \ldots e_n$ basis for N

Let $v_i = \{e_i, \cdot\} \in M$

Assume $v_1 \ldots v_n$ are lin. indep $\quad \sigma = \sum_i \mathbb{R}_{\geq 0} v_i$

$\mathfrak{D} = \{(e_i^\perp, 1 + z^{m_i}) \mid 1 \leq i \leq n\}$

$\rightsquigarrow S(\mathfrak{D})$

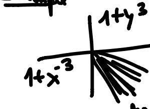
all the automorphisms live in a smaller Lie grp w. Lie algebra

$$\mathfrak{g}' = \bigoplus_{\substack{a_1 \ldots a_i \geq 0 \\ \text{not all} = 0}} k\, z^{\sum_i a_i v_i} \partial_{\sum_i a_i e_i}$$

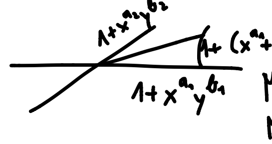
[Mark shows picture from S. Goncharov's paper like , says its a punctured torus, points out that shaded regions are a simplification of what it should be]

<u>Example</u>

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$



$1 + \sum c_k z^k = \prod_{k \geq 1} (1 + z^k)^{a_k}$ $\quad c_k \in \mathbb{Z}$

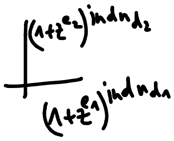


$1 + (x^{a_1} y^{b_1})^{k_1} (x^{a_2} y^{b_2})^{k_2} c_{k_1 k_2}$ $\quad a_k \in \mathbb{Z}$

$M = \mathbb{Z}^2$ $\quad$ Induction on $k_1 + k_2$

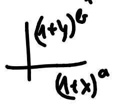
$M' = \langle (a_1, b_1), (a_2, b_2) \rangle \subseteq M$

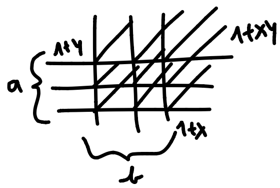
$N' = \mathrm{Hom}(M', \mathbb{Z}) \supseteq N$



As scattering diagram in M'

$n_{d_1}, n_{d_2} \in N \subseteq N'$





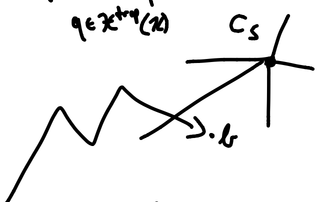
Sean's corollaries

$U = A$ or $\mathcal{X}$ $\quad V = \bigoplus_{q \in \mathcal{X}^{trop}(\mathbb{Z})} k\theta_q$

$U^{trop}(\mathbb{Z}) = N_s$

$\cup$

B $\quad$ From $a \xrightarrow[\gamma]{} b$



$\Rightarrow G$ is an algebra $\qquad a \xrightarrow[\gamma]{} b$ finite

$G = \bigoplus_{q \in \mathcal{X}} k\theta_q \supset A$